

Graphs and the (co)homology of Lie algebras

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Abstract

In this paper, we develop a diamond graph theory and apply the theory to the (co)homology of the Lie algebra generated by positive systems of the classical semi-simple Lie algebras over the field of complex numbers. As an application, we give the weight decomposition of the diamond Lie algebra with Dynkin graph A_{n+1} and compute the rank of every weight subgraph of it.

The classification of compact simply connected Lie groups over \mathbb{C} is due to the classification of the Dynkin graph of the associated semi-simple Lie algebra. The Dynkin graph determines a unique positive root system and Kostant in [3] tells that the integral cohomology of the Lie subalgebra generated by the positive root system is in 1-1 correspondence with the group ring of the Weyl group of the Lie algebra. But the torsion part of the (co)homology of the Lie algebra generated by a positive root system is also very important. For example, let \mathfrak{A}_n be the Lie algebra generated by the positive root system with Dynkin graph A_n and $\mathfrak{A}_\infty = \cup_n \mathfrak{A}_n$ (a graded Lie algebra). Then, $H_*(\mathfrak{A}_n; \mathbb{Z}_p)$ is the E_2 -term of the spectral sequence induced by the lower central series converging to the homology of the group of integral upper-triangular matrices in [2] and $H^*(\mathfrak{A}_\infty; \mathbb{Z}_p)$ (\mathbb{Z}_p the field of integers modular a prime p) is a direct sum summand of the E_1 -term of May spectral sequence in [7] converging to the cohomology of the Steenrod algebra.

In this paper, we develop a diamond graph theory and study the (co)homology of the Lie algebra generated by a diamond root system. Diamond graphs give more information about

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the torsion part the the (co)homology of these Lie algebras. The main results of this paper is Theorem 2.2 and Theorem 3.5.

1 Graphs and (co)chain complexes

In this paper, all objects are of finite type. Graphs are finite graphs and Abelian groups are finitely generated. \otimes means $\otimes_{\mathbb{Z}}$ with \mathbb{Z} the ring of integers. For a set S , $\mathbb{Z}(S)$ denotes the free Abelian group generated by S . S is called a base of $\mathbb{Z}(S)$.

Recall that a graph is a pair of sets (G, E) such that every element of E is a subset of two elements of G . The elements of G are called vertices of the graph and the elements of E are called edges of the graph. We omit the set of edges and simply denote by G the graph (G, E) . We denote by $[a, b]$ an edge and call a and b neighbors of each other. A graph is finite if the set of vertices is a finite set.

Definition 1.1 For a graph G , a gradation $|\cdot|$ on G is a map $|\cdot|: G \rightarrow \mathbb{Z}$ such that if $[a, b]$ is an edge, then $|a| - |b| = \pm 1$. A graph is called gradable if there is a gradation on it. A graded graph with base graph G is a pair $(G, |\cdot|)$ with G a gradable graph and $|\cdot|$ a gradation on it. $(G, |\cdot|)$ is simply denoted by $|G|$. Two graded graphs are isomorphic if there is a graph isomorphism f that keeps the gradation, i.e., $|f(a)| - |f(b)| = |a| - |b|$ for all a, b . A vertex v of a graded graph $|G|$ is called a top vertex if $|u| = |v| - 1$ for every neighbor u of v . A vertex v of $|G|$ is called a bottom vertex if $|u| = |v| + 1$ for every neighbor u of v .

Notice that a gradable graph has no triangle as a subgraph. In fact, it has no $(2n+1)$ -polygon as a subgraph. That is, there is no $2n+1$ vertices v_0, v_1, \dots, v_{2n} such that $[v_0, v_1], \dots, [v_{2n-1}, v_{2n}], [v_{2n}, v_0]$ are all edges.

Definition 1.2 Let $|G|$ be a graded graph. If v is a top vertex of G , then $|\cdot|_1$ defined by $|v|_1 = |v| - 2$ and $|u|_1 = |u|$ for every other vertex u is also a gradation on G which is called the lowering of $|\cdot|$ by the vertex v . The graded graph $|G|_1$ is called the lowering of $|G|$ by the vertex v . If v is a bottom vertex of $|G|$, then $|\cdot|_2$ defined by $|v|_2 = |v| + 2$ and $|u|_2 = |u|$ for

every other vertex u is also a gradation on G which is called the lifting of $|\cdot|$ by the vertex v . The graded graph $|G|_2$ is called the lifting of $|G|$ by the vertex v . Two gradations on G are equivalent if one of them can be obtained from the other by a finite composite of lowerings and liftings. Two graded graphs are equivalent if there is a graph isomorphism that induces a gradation equivalence.

Theorem 1.1 Let G be a gradable graph. Two gradations $|\cdot|$ and $|\cdot|'$ on G are equivalent if and only if $|v| - |v|'$ is even for all $v \in G$.

Proof. The necessary part is by definition. Now we prove that if $|w| - |w|'$ is even for all vertices w , then $|\cdot|$ and $|\cdot|'$ are equivalent. Let b_1 be a vertex such that $|b_1| \geq |w|$ for all $w \in G$. If $|b_1| > 1$, then lower $|\cdot|$ by b_1 and we get a new gradation $|\cdot|_1$. Let b_2 be a vertex such that $|b_2|_1 \geq |w|_1$ for all $w \in G$. If $|b_2|_1 > 1$, then lower $|\cdot|_1$ by b_2 and we get a new gradation $|\cdot|_2$. Repeat this process if there is vertex with degree > 1 . Since G is a finite graph, this process will come to an end. That is, there exists an n and gradations $|\cdot|_1, \dots, |\cdot|_n$ such that each $|\cdot|_{i+1}$ is a lowering of $|\cdot|_i$ by the vertex b_{i+1} and for all $v \in G$, $|v|_n \leq 1$. Similarly, by lifting the smallest degree vertex, we get an m and gradations $|\cdot|_{n+1}, \dots, |\cdot|_{n+m}$ such that each $|G|_{n+i+1}$ is a lifting of $|G|_{n+i}$ by the vertex b_{n+i+1} and for all $w \in G$, $|w|_{n+m} = 0$ or 1 . $|\cdot|$ is equivalent to $|\cdot|_{n+m}$. Similarly, $|\cdot|'$ is equivalent to a gradation $|\cdot|'_{s+t}$ such that $|w|'_{s+t} = 0$ or 1 for all $w \in G$. Since $|w| - |w|'$ is even for all $w \in G$, we have $|\cdot|_{n+m} = |\cdot|'_{s+t}$. Thus, $|\cdot|$ is equivalent to $|\cdot|'$. Q.E.D.

Recall that a path from a to b is a sequence of vertices $a = v_0, v_1, \dots, v_{n-1}, v_n = b$ such that either $[v_{i-1}, v_i]$ is an edge, or $v_{i-1} = v_i$ for $i = 1, \dots, n$. The length of the path is the number of edges $[v_{i-1}, v_i]$. The distance $d(a, b)$ between two vertices a and b is the minimum of lengths of all paths from a to b . If there is no path from a to b , we define $d(a, b) = \infty$. A graph is connected if the distance between every pair of its vertices is finite.

Theorem 1.2 Let G be a connected graph with more than one vertex. G is gradable if and only if its vertex set has a unique distance decomposition $G = G_1 \sqcup G_2$ (\sqcup is the disjoint union) such that for all $u, v \in G_i$, $i = 1, 2$, $d(u, v)$ is even and for all $a \in G_1$ and $b \in G_2$, $d(a, b)$ is odd. G_1 and G_2 are called the distance components of G .

Proof. If there is a distance decomposition $G = G_1 \sqcup G_2$, then the gradation $|\cdot|$ defined by $|a| = 0$ for all $a \in G_1$ and $|b| = 1$ for all $b \in G_2$ is a gradation. So G is gradable.

If G is gradable, then from the proof of Theorem 1.1 we have that there are only two equivalent classes of gradations on G represented by the two gradations $|\cdot|_1$ and $|\cdot|_2$ such that $|v|_1 + |v|_2 = 1$ and $|v|_i = 0$ or 1 for all $v \in G$. Then $G_1 = \{v \in G \mid |v|_1 = 0\}$ and $G_2 = \{v \in G \mid |v|_1 = 1\}$ are the distance components. Q.E.D.

Definition 1.3 For a gradable graph G , any gradation $|\cdot|$ satisfying that $|v| = 0$ or 1 for all $v \in G$ is called a representation gradation of G . The set $G_1 = \{v \in G \mid |v| = 0\}$ and $G_2 = \{v \in G \mid |v| = 1\}$ are called the distance components of the representation gradation.

Theorem 1.3 Let $|G|$ be a connected graded graph. If $|G|$ has only one bottom vertex v , then $d(u, v) = |u| - |v|$ for all vertex u . Such a graded graph is called a positive distance graph relative to v . If $|G|$ has only one top vertex v , then $d(u, v) = |v| - |u|$ for all vertex u . Such a graded graph is called a negative distance graph relative to v .

Proof. We only prove the positive distance case. Suppose $|\cdot|$ is a gradation that has only one bottom vertex v . Let $N = \min\{|u| \mid u \in G\}$. If $|u| = N$, then u is a bottom vertex and so $u = v$. This implies that for all $u \in G$, $|u| \geq |v|$ and the equality holds if and only if $u = v$. We use induction on n to prove that $|u| = |v| + n$ if and only if $d(u, v) = n$. If $n = 0, 1$, the conclusion is trivial. Suppose for some $n > 1$, we have $|u'| = |v| + i$ if and only if $d(u', v) = i$ for $i = 0, 1, \dots, n$. Then for $|u| = |v| + n + 1$, the induction hypothesis implies that $d(u, v) > n$. Since u is not a bottom vertex, there is a neighbor w of u such that $|w| = |v| + n$. By the induction hypothesis, $d(w, v) = n$. So $d(u, v) \leq d(u, w) + d(w, v) = n + 1$. Thus, $d(u, v) = n + 1$. The conclusion holds. Q.E.D.

Definition 1.4 Let G be a gradable graph. A connection ν on G is a map $\nu : G \times G \rightarrow \mathbb{Z}$ that satisfies the following conditions.

- 1) $\nu(a, b) = \nu(b, a)$ for all $a, b \in G$.
- 2) $\nu(a, b) \neq 0$ if and only if $[a, b]$ is an edge of G .

Two connections ν, ν' are equivalent if there is a map $e: G \rightarrow \{\pm 1\}$ such that $\nu(a, b) = e(a)e(b)\nu'(a, b)$ for all $a, b \in G$.

A graph with connection is a pair (G, ν) with G a gradable graph and ν a connection on G .

Notice that a connection can be defined on an ungradable graph. But such a map has no representation matrix defined as follows. So we define connection only on gradable graphs.

Definition 1.5 For a graph with connection (G, ν) , its representation matrix $A = (a_{i,j})_{m \times n}$ is defined as follows. If G has only one vertex, $A = (0)_{1 \times 1}$, the 1×1 zero matrix. The global dimension $D(A)$ of A is defined to be 1. If G is connected and has more than one vertices, take a representation gradation $|\cdot|$ of G and suppose v_1, \dots, v_m and w_1, \dots, w_n are the distance components of $|\cdot|$, then $a_{i,j} = \nu(v_i, w_j)$. The global dimension $D(A)$ of A is defined to be $m+n-2r$, where r is the rank of A . If G is not connected, then its representation matrix is the direct sum (see the next definition) of all its connected component representation matrices and the global dimension of A is the sum of the global dimensions of all its connected component representation matrices.

The representation matrix is not unique and depends on the order of distance components and their vertices. Different equivalent connections have different representation matrices. To make the representation matrices unique under equivalences, we have the following definition.

Definition 1.6 Two matrices over \mathbb{Z} are equivalent if one of them can be obtained from the other by a finite composite of the following transformations.

- 1) Permute the rows of the matrix.
- 2) Replace a row α of the matrix by $-\alpha$.
- 3) Replace the matrix A by its transpose matrix A^T .
- 4) Replace matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $\begin{pmatrix} A^T & 0 \\ 0 & B \end{pmatrix}$.

For two matrices $A = (a_{i,j})_{m_1 \times n_1}$ and $B = (b_{k,l})_{m_2 \times n_2}$, their direct sum is the matrix

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{(m_1+m_2) \times (n_1+n_2)}$$

and their orthogonal product $A \times B = (c_{s,t})_{m \times n}$ is the following matrix. $m = m_1 m_2 + n_1 n_2$, $n = m_1 n_2 + n_1 m_2$, $c_{s,t} = 0$ except the following,

$$\begin{aligned} c_{im_1+k, jn_1+k} &= a_{i,j}, & i &= 1, \dots, m_1, j = 1, \dots, n_1, k = 1, \dots, n_2 \\ c_{m_1 n_2 + jn_1 + k, m_1 m_2 + im_1 + k} &= a_{i,j}, & i &= 1, \dots, m_1, j = 1, \dots, n_1, k = 1, \dots, m_2 \\ c_{m_1 n_2 + kn_1 + i, kn_1 + j} &= b_{i,j}, & i &= 1, \dots, m_2, j = 1, \dots, n_2, k = 1, \dots, n_1 \\ c_{km_1 + j, m_1 m_2 + km_1 + i} &= -b_{i,j}, & i &= 1, \dots, m_2, j = 1, \dots, n_2, k = 1, \dots, m_1 \end{aligned}$$

It is obvious that if A and B are respectively equivalent to A' and B' , $A \oplus B$ is equivalent to $A' \oplus B'$ and $A \times B$ is equivalent to $A' \times B'$.

Definition 1.7 A chain graph is a pair $(|G|, \nu)$ with $|G|$ a graded graph and ν a connection on G such that for every pair $a, b \in G$ with $|b| = |a| + 2$, $\sum_{|c|=|a|+1} \nu(a, c)\nu(c, b) = 0$. Two chain graphs $(|G_1|_1, \nu_1)$ and $(|G_2|_2, \nu_2)$ are similar if there is a graph isomorphism $\psi: G_1 \rightarrow G_2$ such that the induced map $\nu'_1(a, b) = \nu_2(\psi(a), \psi(b))$ for all $a, b \in G_1$ is a connection equivalent to ν_1 . If ψ is a graded graph isomorphism, $(|G_1|_1, \nu_1)$ and $(|G_2|_2, \nu_2)$ are isomorphic.

Definition 1.8 For a chain graph $(|G|, \nu)$, the associated chain complex $(\mathbb{Z}(G), d)$ and associated cochain complex $(\mathbb{Z}(G), \delta)$ of the graph is defined as follows. $dv = \sum_{|w|=|v|-1} \nu(v, w)w$ and $\delta v = \sum_{|w|=|v|+1} \nu(v, w)w$ for all $v \in G$.

The homology of $(\mathbb{Z}(G) \otimes R, d)$ with R a commutative ring is called the homology of $(|G|, \nu)$ over the coefficient ring R and is denoted by $H_*(|G|; R)$ and $H_*(|G|) = H_*(|G|; \mathbb{Z})$. Dually, $H^*(|G|; R) = H^*(\text{Hom}_{\mathbb{Z}}(\mathbb{Z}(G), R), \delta)$ is the cohomology of $(|G|, \nu)$ over the coefficient ring R and $H^*(|G|) = H^*(\mathbb{Z}(G), \delta) = H^*(|G|; \mathbb{Z})$.

Definition 1.9 Let $(|G_1|_1, \nu_1)$ and $(|G_2|_2, \nu_2)$ be two chain graphs. The disjoint union graph $(|G_1 \sqcup G_2|, \nu)$ is defined as follows. The restriction of $|\cdot|$ on G_i is $|\cdot|_i$ and the restriction of ν on $G_i \times G_i$ is ν_i and $\nu(a, b) = 0$ for all $a \in G_1$ and $b \in G_2$. The product graph $(|G_1 \times G_2|, \nu)$ is defined as follows. $|(g_1, g_2)| = |g_1|_1 + |g_2|_2$ for all $g_i \in G_i$. $[(g_1, g_2), (g'_1, g'_2)]$ is an edge of $G_1 \times G_2$ if either $[g_1, g'_1]$ is an edge G and $g_2 = g'_2$, or $g_1 = g'_1$, $[g_2, g'_2]$ is an edge G_2 . For all $g_i, g'_i \in G_i$, $\nu((g_1, g_2), (g'_1, g'_2)) = \nu_1(g_1, g'_1)$, $\nu((g_1, g_2), (g_1, g'_2)) = (-1)^{|g_1|_1} \nu_2(g_2, g'_2)$.

Theorem 1.4 Let $(|G|, \nu)$ and $(|H|, \nu)$ be two chain graphs. The associated chain complex $(\mathbb{Z}(G \sqcup H), d)$ of the disjoint union graph $(|G \sqcup H|, \nu)$ is the direct sum chain complex $(\mathbb{Z}(G) \oplus \mathbb{Z}(H), d)$. The associated chain complex $(\mathbb{Z}(G \times H), d)$ of the product graph $(|G \times H|, \nu)$ is the tensor product chain complex $(\mathbb{Z}(G) \otimes \mathbb{Z}(H), d)$. The same conclusion holds for associated cochain complexes.

Proof. Direct checking.

Q.E.D.

Theorem 1.5 Let $(|G|, \nu)$ be a chain graph. Then

$$D(|G|, \nu) = \sum_{k=-\infty}^{\infty} \dim H_k(|G|) = \sum_{k=-\infty}^{\infty} \dim H^k(|G|) = D(A)$$

for any representation matrix A of the graph with connection (G, ν) , where $\dim H$ means the dimension of the free part of the (co)homology. Thus, the global dimension $D(G, \nu) = D(|G|, \nu)$ of the graph with connection (G, ν) is well-defined and for two similar chain graphs $(|G_1|_1, \nu_1)$ and $(|G_2|_2, \nu_2)$, $D(|G_1|_1, \nu_1) = D(|G_2|_2, \nu_2) = D(G_1, \nu_1)$.

Proof. We may suppose G is connected and only prove the homology case. If $|\cdot|$ is a representation gradation, the conclusion is obvious. Suppose $|\cdot|$ is not a representation gradation and we may suppose the associated chain complex (C, d) of the chain graph (G, ν) is the following.

$$0 \rightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_0} C_0 \rightarrow 0$$

Define chain complex (C', d)

$$0 \rightarrow C_{n-1} \xrightarrow{d'_{n-1}} C_{n-2} \oplus C_n \xrightarrow{d'_{n-2}} \cdots \xrightarrow{d'_0} C_0 \rightarrow 0$$

as follows. Suppose C_n has a base a_1, \dots, a_s and C_{n-1} has a base b_1, \dots, b_t and $d_n(a_i) = \sum_j c_{i,j} b_j$. Then $d'_{n-1}(a_i) = 0$ and $d'_{n-1}(b_i) = d_{n-1}(b_i) + \sum_k c_{k,i} a_k$ and $d'_k = d_k$ for $k < n-2$. It is obvious that (C', d) is the associated chain complex of another chain graph $(|G'|, \nu)$. Since $D(|G|) = \sum_{k=-\infty}^{\infty} \dim H_k(|G|; F)$ for all field F of characteristic 0, we compute the number for $F = \mathbb{R}$, the real number. In this case, we may suppose all the C_n are vector spaces over \mathbb{R} . Let $\delta_n = d'_{n-1} - d_{n-1}$ and $A = (c_{i,j})_{s \times t}$. AA^T is a symmetric matrix and there is an orthogonal matrix Q such that $AA^T = Q^T D Q$ with $D = \text{diag}(k_1, \dots, k_r, 0, \dots, 0)$ a diagonal matrix ($k_i > 0$). Suppose $Q = (q_{i,j})_{s \times s}$ and let $a'_i = \sum_j q_{i,j} a_j$. Then a'_1, \dots, a'_s is another

base of C_n such that if $d_n(a'_i) = \sum_j c'_{i,j} b_j$, then $\delta_n(b_i) = \sum_k c'_{k,i} a'_k$ and that $\delta_n d_n(a'_i) = k_i a'_i$ for $i = 1, \dots, r$, $\delta_n d_n(a'_j) = 0$ for $j = r+1, \dots, s$. By definition, $d_n(a'_j) = 0$ for $j = r+1, \dots, s$. So $\dim H_n(C; \mathbb{R}) = s-r$, $\dim H_n(C'; \mathbb{R}) = 0$, $\dim H_{n-1}(C; \mathbb{R}) = \dim H_{n-1}(C'; \mathbb{R})$, $\dim H_{n-2}(C'; \mathbb{R}) = \dim H_{n-2}(C; \mathbb{R}) + s-r$, $\dim H_k(C; \mathbb{R}) = \dim H_k(C'; \mathbb{R})$ otherwise. Thus, (C, d) and (C', d) have the same global dimension. Repeat the above process to (C', d) and we can prove that (C, d) and the associated chain complex of the representation gradation have the same global dimension. Q.E.D.

Definition 1.10 Let G be a gradable graph. A connection ν on G is deformable if it satisfies that for all $a, b \in G$ with $d(a, b) = 2$, $\sum_{c \in G} \nu(a, c) \nu(c, b) = 0$.

A gradable graph is deformable if there is a deformable connection on it. A deformation graph is a pair (G, ν) with G a gradable graph and ν a deformable connection on G . Two deformation graphs are isomorphic if there is a graph isomorphism that induces a deformable connection equivalence.

Theorem 1.6 For a deformation graph (G, ν) , all its representation matrices are equivalent $n \times n$ square matrices. If ν' is a deformable connection equivalent to ν , all the representation matrices of (G, ν') are equivalent to that of (G, ν) . The equivalent matrix class is called the representation class of (G, ν) . The representation class of the disjoint union of two graphs is the direct sum class of the two graphs. The representation class of the product of two graphs is the orthogonal product class of the two graphs.

Proof. We only prove that representation matrices are square matrices. Other conclusions are direct checkings. We may suppose G is connected with more than one vertices. ν is a deformable connection implies that every representation matrix A satisfies that AA^T and $A^T A$ are both diagonal matrices with positive diagonal entries. This implies that A is a square matrix. Q.E.D.

Definition 1.11 The rank $r(v)$ of a vertex v of a deformation graph (G, ν) is the non-negative integer $r(v) = \sum_{w \in G} \nu(w, v)^2$.

Theorem 1.7 For a connected deformation graph (G, ν) , all its vertex v have the same rank which is called the rank of (G, ν) and is denoted by $r(G, \nu)$.

Proof. If G has only one vertex, then by definition $r(G, \nu) = 0$. Suppose G has more than one vertices and v_1, \dots, v_n and w_1, \dots, w_n are the distance components of G with representation matrix $A = (a_{i,j} = \nu(v_i, w_j))_{n \times n}$. Then $AA^T = \text{diag}(d_1, \dots, d_n)$ and $A^T A = \text{diag}(d'_1, \dots, d'_n)$ with $d_i = r(v_i)$ and $d'_j = r(w_j)$, where $\text{diag}(\dots)$ represents the diagonal square matrix. Thus, $AA^T A = \text{diag}(d_1, \dots, d_n)A = A \text{diag}(d'_1, \dots, d'_n)$ and $a_{i,j} \neq 0$ implies $d_i = d'_j$. Since G is connected, for any i, j , there is a path $v_i, w_{j_1}, v_{i_1}, \dots, w_{j_s}, v_{i_s}, w_j$ from v_i to w_j . So $a_{i,j_1}, a_{i_1,j_1}, a_{i_1,j_2}, a_{i_2,j_2}, \dots, a_{i_s,j_s}, a_{i_s,j}$ are all non-zero and $d_i = d'_{j_1} = d_{i_1} = d'_{j_2} = \dots = d'_{j_s} = d_{i_s} = d'_j$. Similarly, there is a path from v_i to v_j for $i \neq j$ and we have $d_i = d_j$. So $AA^T = A^T A = rE$ (E unit matrix) and $r = r(v_i) = r(w_j)$ for all i, j . Q.E.D.

Theorem 1.8 Let (G, ν) be a deformation graph. Then for any gradation $|\cdot|$ on G , $(|G|, \nu)$ is a chain graph. Such a chain graph is called a deformable chain graph.

Proof. ν is a deformable connection implies that for any gradation $|\cdot|$ on G and $a, b \in G$ with $|b| - |a| = \pm 2$, $\sum_{|c|=|b|+1} \nu(a, c)\nu(c, b) = \sum_{c \in G} \nu(a, c)\nu(c, b) = 0$. Q.E.D.

Theorem 1.9 For a connected deformable chain graph $(|G|, \nu)$ with rank > 0 , both $H_*(|G|)$ and $H^*(|G|)$ are torsion groups and $H^k(|G|) = H_{k-1}(|G|)$ for all k .

Proof. Since the representation matrix A of G is an orthogonal matrix and so its global dimension $D(A) = 0$. By Theorem 1.5, $D(G, \nu) = 0$ and so the free part of $H_k(|G|)$ and $H^k(|G|)$ are trivial. By universal coefficient theorem, $H^k(|G|) = \text{Ext}(H_{k-1}(|G|), \mathbb{Z}) = H_{k-1}(|G|)$ for all k . Q.E.D.

Theorem 1.10 Let $(|G|_1, \nu)$ be a connected deformable chain graph with rank $n > 0$. If v is a bottom vertex of $|G|_1$ with $|v|_1 = q$ and $|G|_2$ is the lifting of $|G|_1$ by v , then there is a divisor k of n such that $(\langle m \rangle$ denotes the group of integers modular m)

$$\begin{aligned} H_q(|G|_1)/\langle k \rangle &= H_q(|G|_2), \\ H_{q+1}(|G|_2)/H_{q+1}(|G|_1) &= \langle n/k \rangle, \\ H_i(|G|_1) &= H_i(|G|_2) \text{ if } i \neq q, q+1. \end{aligned}$$

Dually, if v is a top vertex with $|v|_1 = q$ and $|G|_2$ is the lowering of $|G|_1$ by v , then there is a divisor k of n such that

$$\begin{aligned} H^q(|G|_1)/\langle k \rangle &= H^q(|G|_2), \\ H^{q-1}(|G|_2)/H^{q-1}(|G|_1) &= \langle n/k \rangle, \\ H^i(|G|_1) &= H^i(|G|_2) \text{ if } i \neq q, q-1. \end{aligned}$$

Proof. We only prove the cohomology case. Suppose $(C_i, \delta_i, |\cdot|_i)$ is the associated cochain complex of $(|G|_i, \nu)$. It is obvious that the free Abelian group generated by v is a cochain subcomplex of C_1 and that the free Abelian group generated by vertices other than v is a cochain subcomplex of C_2 . Denote the first cochain subcomplex of C_1 by T_1 , then the quotient cochain complex C_1/T_1 is just the second cochain subcomplex of C_2 generated by vertices other than v . Denote C_1/T_1 by \tilde{C} and C/\tilde{C} by T_2 . Notice that $H^{q-2}(T_2) = H^q(T_1) = \mathbb{Z}$; $H^s(T_2) = H^t(T_1) = 0$, otherwise. From the two short exact sequences of cochain complexes $0 \rightarrow \tilde{C} \rightarrow C_2 \rightarrow T_2 \rightarrow 0$ and $0 \rightarrow T_1 \rightarrow C_1 \rightarrow \tilde{C} \rightarrow 0$, we have two exact sequences of Abelian groups

$$\begin{aligned} 0 \rightarrow H^{q-2}(\tilde{C}) \rightarrow H^{q-2}(C_2) \rightarrow H^{q-2}(T_2) &\xrightarrow{\tau} H^{q-1}(\tilde{C}) \rightarrow H^{q-1}(C_2) \rightarrow 0 \\ 0 \rightarrow H^{q-1}(C_1) \rightarrow H^{q-1}(\tilde{C}) &\xrightarrow{\pi} H^q(T_1) \rightarrow H^q(C_1) \rightarrow H^q(\tilde{C}) \rightarrow 0 \end{aligned}$$

and have that $H^s(C_2) = H^s(\tilde{C})$ if $s \neq q-1, q-2$ and that $H^s(C_1) = H^s(\tilde{C})$ if $s \neq q-1, q$. So the two exact sequence are

$$\begin{aligned} 0 \rightarrow H^{q-2}(C_1) \rightarrow H^{q-2}(C_2) \rightarrow \mathbb{Z} &\xrightarrow{\tau} H^{q-1}(\tilde{C}) \rightarrow H^{q-1}(C_2) \rightarrow 0 \quad (*) \\ 0 \rightarrow H^{q-1}(C_1) \rightarrow H^{q-1}(\tilde{C}) &\xrightarrow{\pi} \mathbb{Z} \rightarrow H^q(C_1) \rightarrow H^q(C_2) \rightarrow 0 \quad (**) \end{aligned}$$

Since $H^q(C_1)$ is a torsion group, we have that π is an epimorphism. Thus, there is an integer k such that $(**)$ implies two short exact sequences

$$0 \rightarrow H^{q-1}(C_1) \rightarrow H^{q-1}(\tilde{C}) \xrightarrow{\pi} \mathbb{Z} \rightarrow 0, \quad 0 \rightarrow \langle k \rangle \rightarrow H^q(C_1) \rightarrow H^q(C_2) \rightarrow 0$$

Since $H^{q-1}(C_1)$ is a torsion group, we have that $H^{q-1}(\tilde{C}) = H^{q-1}(C_1) \oplus \mathbb{Z}$ and the generator x of the free subgroup of $H^{q-1}(\tilde{C})$ satisfies $\pi(x) = k[v]$, where $[v]$ denote the cohomology class represented by v .

Since $H^{q-2}(C_2)$ is a torsion group, $(*)$ implies two short exact sequences

$$0 \rightarrow H^{q-2}(C_1) \rightarrow H^{q-2}(C_2) \rightarrow 0, \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\tau} H^{q-1}(C_1) \oplus \mathbb{Z} \rightarrow H^{q-1}(C_2) \rightarrow 0$$

Since $H^{q-1}(C_2)$ is a torsion group, we have that $H^{q-1}(C_1)$ is a subgroup of $H^{q-1}(C_2)$ and $H^{q-1}(C_2)/H^{q-1}(C_1) = (H^{q-1}(C_1) \oplus \mathbb{Z})/(H^{q-1}(C_1) + \text{im}\tau) = \langle k' \rangle$ and k' satisfies that $\tau([v]) \equiv k'x \pmod{H^{q-1}(C_1)}$. Thus, $\pi\tau([v]) = \pi(k'x) = kk'[v]$. Suppose $\delta_2 v = \sum_{i=1}^n \nu(v, v_i)v_i$, then $\delta_1 v_i = \nu(v, v_i)v + \delta_2 v_i$ for $i = 1, \dots, n$, so

$$\pi\tau([v]) = [\delta_1 \delta_2 v] = [\delta_1 (\sum_{i=1}^n \nu(v, v_i)v_i)] = [nv + \delta_2 (\sum_{i=1}^n \nu(v, v_i)v_i)] = [nv + \delta_2^2 v] = n[v].$$

Thus, $k' = n/k$.

Q.E.D.

Definition 1.12 For a connected deformation graph (G, ν) , its volume is defined as follows. If G has only one vertex, its volume is 0. If G has more than one vertices, its volume is the number of vertices of one of its distance components. The characteristic number of (G, ν) is $\chi(G, \nu) = |\det A|$, the absolute value of the determinant of any of its representation matrix A . For a connected deformable chain graph $(|G|, \nu)$ with $\text{rank} > 0$, its characteristic number is

$$\chi(|G|, \nu) = \frac{\prod_{n=-\infty}^{+\infty} |H_{2n}(|G|)|}{\prod_{n=-\infty}^{+\infty} |H_{2n+1}(|G|)|} = \frac{\prod_{n=-\infty}^{+\infty} |H^{2n+1}(|G|)|}{\prod_{n=-\infty}^{+\infty} |H^{2n}(|G|)|},$$

where $|H|$ denotes the cardinality of the finite group H .

Theorem 1.11 For a connected deformation graph (G, ν) with volume n and $r(G, \nu) = r$,

$$\chi(G, \nu)^2 = r^n.$$

Specifically, when n is odd, $r(G, \nu)$ is a square number.

For a connected deformable chain graph $(|G|, \nu)$ with $\text{rank } r > 0$,

$$\chi(|G|, \nu) = \chi(G, \nu)r^\mu,$$

where $\mu = \sum_{k=0}^{+\infty} k(\mu_{2(k+s)+1} - \mu_{2(k+s)})$ and μ_k = number of vertices with degree k and s satisfies that $\mu_i = 0$ for $i < 2s$ and $\mu_{2s+1} \neq 0$.

Proof. The first equality is obtained from the equality $AA^T = rE$ of a representation matrix by taking determinant. Let $|\cdot|_1$ be a representation gradation with representation matrix A . By definition, $\chi(|G|_1, \nu) = |H_0(|G|_1)| = |\det A| = \chi(G, \nu)$. Then the second equality of the theorem is a corollary of Theorem 1.10.

Q.E.D.

Theorem 1.12 Let (G_i, ν_i) , $i = 1, 2$ be two connected deformation graph with rank r_i and volume n_i . Then $r(G_1 \times G_2, \nu) = r_1 + r_2$ and $\chi(G_1 \times G_2, \nu) = (r_1 + r_2)^{n_1 n_2}$.

Proof. By definition.

Q.E.D.

Notice that $\chi(G \times H, \nu)$ in the above equality is no longer a square since the volume of the product graph is $2n_1 n_2$.

Theorem 1.13 Let (G, ν) be a connected deformation graph with rank $r > 0$ and F be a field of characteristic p . If $p = 0$ or $p > 0$ but is not a divisor of r , then for all deformable chain graphs $(|G|, \nu)$, $H_*(|G|; F) = 0$ and $H^*(|G|; F) = 0$.

Proof. By Theorem 1.11.

Q.E.D.

Definition 1.13 A finite graph G is called a diamond graph if it has the following property. If there are three vertices a, b, c of G such that $[a, b]$ and $[b, c]$ are edges, then there exists one and only one new vertex d such that $[c, d]$ and $[d, a]$ are edges and neither of $[a, c]$ and $[b, d]$ is an edge. The subgraph consisting of such four vertices and four edges is called a diamond of G . We use four vertices a, b, c, d to denote a diamond such that $[a, b], [b, c], [c, d], [d, a]$ are edges and $[a, c]$ and $[b, d]$ are not edges.

The above definition implies that there is no triangle in a diamond graph.

Theorem 1.14 For a connected diamond graph G , all its vertices have the same number of neighbors which is called the rank of G and is denoted by $r(G)$.

Proof. If G has no edge, then it has only one vertex with rank 0. If a is a vertex of G that has rank 1, then G has only one edge $[a, b]$ and two vertices a and b . The number of neighbors of a and b are all 1 and the conclusion holds. Suppose G has a vertex a with $n > 1$ neighbors. Let b be a neighbor of a and A and B are respectively the set of neighbors of a and b . For any $v \in A - \{b\}$, three vertices v, a, b determine a unique vertex w such that v, a, b, w form a diamond. This obviously sets up a 1-1 correspondence between $A - \{b\}$ and

$B - \{a\}$. So A and B have the same cardinality. Since G is connected, all its vertices have the same number of neighbors. Q.E.D.

Example 1.1 There exist ungradable diamond graphs. For example, let D_1 be the diamond graph with vertex set $\{v, v_i, v_{i,j}=v_{j,i} \mid 1 \leq i < j \leq 5\}$ and the edges $[v, v_i]$, $[v_i, v_{i,j}]$, $[v_{i,j}, v_{s,t}]$ if $\{i, j\} \cap \{s, t\} = \emptyset$. The distance function $|u| = d(u, v)$ is not a gradation, for $|v_{i,j}| = |v_{s,t}| = 2$ but $[v_{i,j}, v_{s,t}]$ is an edge for $\{i, j\} \cap \{s, t\} = \emptyset$. By Theorem 1.3, a distance function of a gradable graph must be a gradation. So D_1 is not gradable.

Definition 1.14 Let G be a gradable diamond graph. A signature ν on G is a deformable connection on G such that $\nu(a, b) = \pm 1$ for all edges $[a, b]$.

An equivalent definition of a signature is that for every diamond, three of the four edges have the same sign of signature and the other edge have the other sign of signature.

Theorem 1.15 Let G be a diamond graph. For any deformable connection ν on G , there is a unique associated signature $\bar{\nu}$ defined by $\nu(a, b) = |\nu(a, b)|\bar{\nu}(a, b)$ for all edges $[a, b]$ and $\bar{\nu}(a, b) = 0$ if $\nu(a, b) = 0$. Two deformable connections ν and ν' are equivalent if and only if their associated signatures $\bar{\nu}$ and $\bar{\nu}'$ are equivalent and $|\nu(a, b)| = |\nu'(a, b)|$ for all a, b . Moreover, all signatures on G if they exist are equivalent. Thus, a gradable diamond graph is deformable if and only if there is a signature on it.

Proof. We only prove the uniqueness of the equivalent class of signatures. Other conclusions are trivial. We may suppose the diamond graph G is connected. Let \tilde{G} be the 2-dimensional CW-complex defined as follows. The 1-skeleton of \tilde{G} is just the graph G with its usual CW-complex structure. To every diamond we associated a 2-cell with attaching map a homeomorphism from S^1 to the four edges of the diamond. Take a maximal tree on the graph G and suppose E_1, \dots, E_n are the edges that is not in the maximal tree. Then $\pi_1(G)$ (regard G as a CW-complex) is the free group generated by any n loops that successively containing only one edge E_i for $i = 1, \dots, n$. Since G is connected, every edge is the edge of a diamond. This implies that $\pi_1(\tilde{G}) = 0$.

Let ν, ν' be two signatures. Take a fixed vertex v of G and for a path $\omega = \{v, v_1, \dots, v_n, u\}$, define $e(\omega, u) = \nu(v, v_1)\nu'(v, v_1)\nu(v_1, v_2)\nu'(v_1, v_2) \cdots \nu(v_n, u)\nu'(v_n, u)$ ($\nu(a, a) = \nu'(a, a) = 1$). It is a direct checking that if two paths ω_1 and ω_2 differ only on a diamond, $e(\omega_1, u) = e(\omega_2, u)$. Thus e is invariant on homotopic loops in \tilde{G} . Since $\pi_1(\tilde{G}) = 0$, $e(u, \omega)$ only depends on the end vertex u and so $e(u) = e(\omega, u)$ is well-defined. So for any $a, b \in G$, $\nu(a, b) = e(a)e(b)\nu'(a, b)$. ν and ν' are equivalent. Q.E.D.

Example 1.2 There exist gradable diamond graphs that has no signature. Let D_2 be the diamond graph with vertex set $\{v, v_i, v_{i,j} = v_{j,i}, u_1, \dots, u_6 \mid 1 \leq i < j \leq 5\}$ and gradation $|v| = 0$, $|v_i| = 1$, $|v_{i,j}| = 2$, $|u_k| = 3$. The edges are $[v, v_i]$, $[v_i, v_{i,j}]$, and

$$\begin{aligned} &[u_1, v_{1,2}], [u_1, v_{2,3}], [u_1, v_{3,4}], [u_1, v_{4,5}], [u_1, v_{5,1}], [u_2, v_{1,2}], [u_2, v_{2,4}], [u_2, v_{4,5}], [u_2, v_{5,3}], [u_2, v_{3,1}], \\ &[u_3, v_{1,2}], [u_3, v_{2,5}], [u_3, v_{5,3}], [u_3, v_{3,4}], [u_3, v_{4,1}], [u_4, v_{3,2}], [u_4, v_{2,4}], [u_4, v_{4,1}], [u_4, v_{1,5}], [u_4, v_{5,3}], \\ &[u_5, v_{3,2}], [u_5, v_{2,5}], [u_5, v_{5,4}], [u_5, v_{4,1}], [u_5, v_{1,3}], [u_6, v_{4,2}], [u_6, v_{2,5}], [u_6, v_{5,1}], [u_6, v_{1,3}], [u_6, v_{3,4}]. \end{aligned}$$

The volume of D_2 is 11 but the rank is 5 and not a square number. So by Theorem 1.11, D_2 has no deformable connection on it.

Definition 1.15 A gradable diamond graph G that has a signature is called admissible. For a gradation $|\cdot|$ on G , the deformable chain graph $(|G|, \nu)$ with ν any of its signature is called a GAD (graded, admissible, diamond) graph. Since there is only one equivalent class of signatures, $(|G|, \nu)$ is often simply denoted by $|G|$.

It is obvious that the rank of a connected GAD graph $|G|$ as a deformable chain graph equals the rank of the diamond graph as defined in Theorem 1.14.

Example 1.3 The (co)homology of the distance graph of an admissible diamond graph may not be trivial. Let (C, d) be defined as follows. The vertex set is $\{v, v_i, v_{i,j} = -v_{j,i}, 1 \leq i < j \leq 4, e_1, e_2, e_3\}$, $|v| = 0$, $|v_i| = 1$, $|v_{i,j}| = 2$, $|e_k| = 3$, $dv = 0$, $dv_i = v$, $dv_{i,j} = v_i - v_j$, $de_1 = v_{1,2} + v_{2,3} + v_{3,4} + v_{4,1}$, $de_2 = v_{1,3} + v_{3,4} + v_{4,2} + v_{2,1}$, $de_3 = v_{1,4} + v_{4,2} + v_{2,3} + v_{3,1}$. It has the positive distance gradation of a diamond graph. $H_2(C) = \langle 2 \rangle$ with generator class represented by $v_{2,3} + v_{3,4} + v_{4,2}$; $H_s(C) = 0$, otherwise.

2 Diamond root systems

Recall that a Lie algebra \mathfrak{G} (over \mathbb{Z}) is an Abelian group with Lie bracket $[\cdot, \cdot]: \mathfrak{G} \otimes \mathfrak{G} \rightarrow \mathfrak{G}$ that satisfies the following properties.

- 1) $[x, x] = 0$ for all $x \in \mathfrak{G}$.
- 2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{G}$.

In this paper, we only study free Lie algebras, i.e., Lie algebras that are free Abelian groups generated by a finite set S which is called the base of the Lie algebra.

Definition 2.1 For a Lie algebra \mathfrak{G} with base S , its associated chain graph $(|\Lambda(S)|, \nu)$ with respect to S is defined as follows. Suppose $S = \{e_1, \dots, e_n\}$ and $[e_i, e_j] = \sum_{k=1}^n r_{i,j}^k e_k$ with $r_{i,j}^k \in \mathbb{Z}$ for $i < j$. Let $\Lambda(\mathfrak{G})$ be the exterior algebra generated by \mathfrak{G} . Then $\Lambda(S)$ is the base of $\Lambda(\mathfrak{G})$ consisting of 1 and elements of the form $e_{i_1} e_{i_2} \cdots e_{i_s}$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ with gradation $|1| = 0$ and $|e_{i_1} e_{i_2} \cdots e_{i_s}| = s$. There are two dual derivatives d and δ on $\Lambda(\mathfrak{G})$. $(\Lambda(\mathfrak{G}), \delta)$ is a DGA such that $\delta e_i = \sum_{s < t} r_{s,t}^i e_s e_t$ for $i = 1, \dots, n$ and $d(e_{i_1} e_{i_2} \cdots e_{i_s}) = \sum_{u < v} (-1)^{v-u} [e_{i_u}, e_{i_v}] e_{i_1} \cdots \hat{e}_{i_u} \cdots \hat{e}_{i_v} \cdots e_{i_s}$. d and δ naturally induce the same connection ν on $|\Lambda(S)|$. Thus, $(|\Lambda(S)|, \nu)$ is the chain graph with respectively associated chain complex and cochain complex $(\Lambda(\mathfrak{G}), d)$ and $(\Lambda(\mathfrak{G}), \delta)$. $\Lambda(S)$ ($|\Lambda(S)|$) is called the (graded) base graph of \mathfrak{G} and ν is called the associated connection of \mathfrak{G} with respect to base S .

Definition 2.2 A diamond Lie algebra \mathfrak{G} is a Lie algebra with base S that satisfies the following property. Such a base S is called a diamond root system.

- 1) For $\alpha \in S$, $-\alpha \notin S$.
- 2) For two different $\alpha, \beta \in S$, either $[\alpha, \beta] = 0$ or $\pm[\alpha, \beta] \in S - \{\alpha, \beta\}$.
- 3) For three different $\alpha, \beta, \gamma \in S$ such that $\pm[\alpha, \beta], \pm[\beta, \gamma] \in S$, $[\alpha, \gamma] = 0$ and $\pm[\alpha, \beta, \gamma] = \pm[[\alpha, \beta], \gamma] = \pm[\alpha, [\beta, \gamma]] \in S$ and $[\alpha, \beta] \neq \pm[\beta, \gamma]$. Such three different α, β, γ are called adjacent.

- 4) For four different $\xi, \eta, \sigma, \tau \in S$ such that $[\xi, \eta] = [\sigma, \tau] \in S$, there are adjacent $\alpha, \beta, \gamma \in S$ such that $\xi = \alpha$, $\eta = \pm[\beta, \gamma]$, $\sigma = \pm[\alpha, \beta]$, $\tau = \gamma$ (permute ξ, η, σ, τ if necessary) and there are no adjacent $\alpha', \beta', \gamma' \in S$ such that $\eta = \alpha'$, $\xi = \pm[\beta', \gamma']$.

Theorem 2.1 For a Lie algebra \mathfrak{g} with diamond root system S , its base graph $\Lambda(S)$ is a diamond graph and the associated chain graph $(|\Lambda(S)|, v)$ is a GAD graph.

Proof. We will prove that for a diamond root system S , $\Lambda(S)$ is a diamond graph with only the following six types of diamond ($[\alpha, \beta, \gamma] = [[\alpha, \beta], \gamma] = [\alpha, [\beta, \gamma]]$ and an arrow represents an edge from the vertex with higher degree to the vertex with lower degree)

$$\begin{array}{lcl}
(1) & \begin{array}{c} \xi\eta\sigma\tau x \\ \swarrow \quad \searrow \\ [\xi, \eta]\sigma\tau x \quad \xi\eta[\sigma, \tau]x \\ \searrow \quad \swarrow \\ [\xi, \eta][\sigma, \tau]x \end{array} & \\
(2) & \begin{array}{c} \alpha\beta\gamma x \\ \swarrow \quad \searrow \\ \alpha[\beta, \gamma]x \quad \gamma[\alpha, \beta]x \\ \searrow \quad \swarrow \\ [\alpha, \beta, \gamma]x \end{array} & \\
(3) & \begin{array}{c} \alpha\beta\gamma[\alpha, \beta]x \\ \swarrow \quad \searrow \\ \alpha[\alpha, \beta][\beta, \gamma]x \quad \alpha\beta[\alpha, \beta, \gamma]x \\ \searrow \quad \swarrow \\ [\alpha, \beta][\alpha, \beta, \gamma]x \end{array} & \\
(4) & \begin{array}{c} \alpha\beta\gamma[\alpha, \beta][\beta, \gamma]x \\ \swarrow \quad \searrow \\ \alpha\beta[\beta, \gamma][\alpha, \beta, \gamma]x \quad \beta\gamma[\alpha, \beta][\alpha, \beta, \gamma]x \\ \searrow \quad \swarrow \\ [\alpha, \beta][\beta, \gamma][\alpha, \beta, \gamma]x \end{array} & \\
(5) & \begin{array}{c} \alpha\beta[\beta, \gamma]x \quad \beta\gamma[\alpha, \beta]x \\ \downarrow \quad \searrow \quad \swarrow \quad \downarrow \\ [\alpha, \beta][\beta, \gamma]x \quad \beta[\alpha, \beta, \gamma]x \end{array} & \\
(6) & \begin{array}{c} \alpha\gamma[\alpha, \beta][\beta, \gamma]x \quad \alpha\beta\gamma[\alpha, \beta, \gamma]x \\ \downarrow \quad \searrow \quad \swarrow \quad \downarrow \\ \alpha[\beta, \gamma][\alpha, \beta, \gamma]x \quad \gamma[\alpha, \beta][\alpha, \beta, \gamma]x \end{array} &
\end{array}$$

where x is a product of elements of S with no factors appearing in the front.

The uniqueness of the above six types of diamonds is from the definition of the diamond system.

In the following discussion, sign is neglected. That is, $x = y$ implies $x = \pm y$. Thus, we may suppose the root system satisfies that $[\alpha, \beta] = \alpha + \beta$ for all roots $\alpha, \beta, \alpha + \beta$.

For three $u, v, w \in \Lambda(S)$ such that u and v are neighbors and v and w are neighbors, they are one of the following, where $\xi, \eta, \sigma, \tau \in S$ and $x, y \in \Lambda(S)$.

- 1) $u = [\xi, \eta]x, v = \xi\eta x = [\sigma, \tau]y, w = \sigma\tau y;$
- 2) $u = [\xi, \eta]x, v = \xi\eta x = \sigma\tau y, w = [\sigma, \tau]y;$
- 3) $u = \xi\eta x, v = [\xi, \eta]x = [\sigma, \tau]y, w = \sigma\tau y.$

If the six elements $\xi, \eta, \sigma, \tau, [\xi, \eta], [\sigma, \tau]$ of S are different and so x and y have none of the six factors, then all the above cases are three vertices of the diamond of type (1). If there are repetitions in the six elements, then from the symmetry of the pairs ξ, η and σ, τ , we need only prove the following four cases. (a) $[\xi, \eta] = [\sigma, \tau];$ (b) $\eta = [\sigma, \tau];$ (c) $\tau = [\xi, \eta];$ (d) $\xi = \sigma.$

(a) In this case, there are three $\alpha, \beta, \gamma \in S$ such that $\xi = \alpha, \eta = [\beta, \gamma], \sigma = [\alpha, \beta], \tau = \gamma.$ If u, v, w are case 1), then $v = \xi\eta[\sigma, \tau]z$ and so $u = [\xi, \eta][\sigma, \tau]z = 0.$ Impossible. If u, v, w are case 2), then $u = \gamma[\alpha, \beta][\alpha, \beta, \gamma]z, v = \alpha\gamma[\alpha, \beta][\beta, \gamma]z, w = \alpha[\beta, \gamma][\alpha, \beta, \gamma]z.$ If z has no factor β , then u, v, w are three vertices of a diamond of type (6). If z has factor β , then u, v, w are three vertices of a diamond of type (4). If u, v, w are case 3), then $u = \alpha[\alpha, \beta, \gamma]z, v = [\alpha, \beta, \gamma]z, w = \gamma[\alpha, \beta]z.$ If z has no factor β , then u, v, w are three vertices of a diamond of type (2). If z has factor β , u, v, w are three vertices of a diamond of type (5).

(b) In this case, $\eta = [\sigma, \tau].$ By Jacobi identity, $[\xi, [\sigma, \tau]] + [\sigma, [\tau, \xi]] + [\tau, [\xi, \sigma]] = 0,$ only one of $[\xi, \sigma]$ and $[\xi, \tau]$ is not zero. We may suppose $[\xi, \sigma] \neq 0$ and $[\xi, \tau] = 0.$ Thus, ξ, σ, τ are adjacent. If u, v, w are case 1), then $u = [\xi, \sigma, \tau]z, v = \xi[\sigma, \tau]z, w = \xi\sigma\tau z.$ If z has no factor $[\xi, \sigma],$ then u, v, w are three vertices of a diamond of type (2). If z has factor $[\xi, \sigma],$ u, v, w are three vertices of a diamond of type (3). If u, v, w are case 2), then $v = \xi\eta\sigma\tau z, w = \xi\eta[\sigma, \tau]z = 0.$ Impossible. If u, v, w are case 3), then $v = [\xi, \eta][\sigma, \tau]z, u = \xi\eta[\sigma, \tau]z = 0.$ Impossible.

(c) In this case, $\sigma = [\xi, \eta].$ By Jacobi identity, $[[\xi, \eta], \tau] + [[\eta, \tau], \xi] + [[\tau, \xi], \eta] = 0,$ only one of $[\xi, \tau]$ and $[\eta, \tau]$ is not zero. We may suppose $[\xi, \tau] = 0$ and $[\eta, \tau] \neq 0.$ Thus, ξ, η, τ are adjacent. If u, v, w are case 1), then $u = [\xi, \eta][\xi, \eta, \tau]z, v = \xi\eta[\xi, \eta, \tau]z, w = \xi\eta\tau[\xi, \eta]z.$ If z has no factor $[\eta, \tau],$ then u, v, w are three vertices of a diamond of type (3). If z has factor $[\eta, \tau],$ u, v, w are three vertices of a diamond of type (4). If u, v, w are case 2), then $v = \xi\eta\sigma\tau z, u = [\xi, \eta]\sigma\tau z = 0.$ Impossible. If u, v, w are case 3), then $v = [\xi, \eta][\sigma, \tau]z, w = [\xi, \eta]\sigma\tau z = 0.$ Impossible.

(d) In this case, $\xi = \sigma$, $\eta \neq \tau$ and η, ξ, τ are adjacent. If u, v, w are case 1), then $v = \xi\eta[\xi, \tau]z$ and so $w = \xi\eta\xi\tau z = 0$. Impossible. If u, v, w are case 2), then $u = [\xi, \eta]\tau z$, $v = \xi\eta\tau z$, $w = \eta[\xi, \tau]z$. If z has no factor $[\eta, \xi, \tau]$, then u, v, w are three vertices of a diamond of type (2). If z has factor $[\eta, \xi, \tau]$, then u, v, w are three vertices of a diamond of type (6). If u, v, w are case 3), then $u = \xi\eta[\xi, \tau]z$, $v = [\xi, \eta][\xi, \tau]z$, $w = \xi\tau[\eta, \tau]z$. If z has no factor $[\eta, \xi, \tau]$, then u, v, w are three vertices of a diamond of type (5). If z has factor $[\eta, \xi, \tau]$, u, v, w are three vertices of a diamond of type (4).

Overall, the types of diamond corresponding to case 1),2),3) and (a),(b),(c),(d) are as in the following table.

	1)	2)	3)
(a)	impossible	(4) or (6)	(2) or (5)
(b)	(2) or (3)	impossible	impossible
(c)	(3) or (4)	impossible	impossible
(d)	impossible	(2) or (6)	(4) or (5)

Q.E.D.

Theorem 2.2 Let \mathfrak{g} be the Lie algebra (over \mathbb{Z}) generated by the positive root system of a simple Lie algebra over \mathbb{C} . Except the case when the Dynkin graph is C_n ($n > 2$) or F_4 , \mathfrak{g} is a diamond Lie algebra.

Proof. Direct checkings.

Q.E.D.

The chain graph $(|\Lambda(\mathfrak{g})|, \nu)$ when the Dynkin graph is C_n or F_4 is even not a deformable chain graph.

For a diamond root system S , give S an order and define $w(S)$ to be the Abelian group generated by S modular zero relations $[\alpha, \beta] = \alpha + \beta$ for all $\alpha < \beta$ and $[\alpha, \beta] \neq 0$. A tedious proof shows that if $w(S)$ is a free Abelian group generated by T such that T is a connected graph ($[a, b]$ is an edge of T if and only if $a + b \in S$), then S is one of those in Theorem 2.2.

Definition 2.3 Let \mathfrak{g} be a diamond Lie algebra with base S . Let $\omega(S)$ denote the set of connected components of the base graph $\Lambda(S)$. Then there is a graph connected component decomposition $\Lambda(S) = \sqcup_{\alpha \in \omega(S)} \Lambda(\alpha)$ and corresponding chain complex and cochain complex

decompositions

$$(\Lambda(\mathfrak{G}), d) = \oplus_{\alpha \in \omega(S)} (\Lambda(\alpha), d), \quad (\Lambda(\mathfrak{G}), \delta) = \oplus_{\alpha \in \omega(S)} (\Lambda(\alpha), \delta).$$

Denote $H_{\alpha,*}(\mathfrak{G}) = H_*(\Lambda(\alpha), d)$ and $H^{\alpha,*}(\mathfrak{G}) = H^*(\Lambda(\alpha), \delta)$, then there is a direct sum decomposition

$$H_*(\mathfrak{G}) = \oplus_{\alpha \in \omega(S)} H_{\alpha,*}(\mathfrak{G}), \quad H^*(\mathfrak{G}) = \oplus_{\alpha \in \omega(S)} H^{\alpha,*}(\mathfrak{G}).$$

Theorem 2.3 Let \mathfrak{G} be a diamond Lie algebra with base S and for $\alpha \in \omega(S)$, $r(\alpha)$ denote the rank of the connected diamond graph $\Lambda(\alpha)$. Then the free part of the (co)homology group of \mathfrak{G} is the free Abelian group generated by all $\alpha \in \omega(S)$ with $r(\alpha) = 0$. For a prime p , if $r(\alpha)$ is not divisible by p , then $H_{\alpha,*}(\mathfrak{G})$ and $H^{\alpha,*}(\mathfrak{G})$ have no p -torsion part.

Proof. By Theorem 1.13.

Q.E.D.

3 Weight (co)chain subcomplexes of \mathfrak{A}_{n+1}

Let \mathfrak{A}_{n+1} be the Lie algebra generated by the positive root system with Dynkin graph A_{n+1} . We will determine all the connected components of $\Lambda(\mathfrak{A}_{n+1})$ and compute their rank. Precisely, let \mathfrak{A}_{n+1} be the Lie algebra of all $(n+1) \times (n+1)$ upper triangular matrices with integer entries and diagonal zero. For simplicity, we denote the entries of a $(n+1) \times (n+1)$ matrix by $a_{i,j}$ with $i, j = 0, 1, \dots, n$. Let $\{e_{i,j} \mid 0 \leq i < j \leq n\}$ denote the matrix with all entries 0 except $a_{i,j} = 1$, then the Lie bracket of the Lie algebra is defined by

$$[e_{i,j}, e_{k,l}] = \begin{cases} e_{i,l} & \text{if } j = k \\ -e_{j,k} & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}.$$

We denote the associated chain and cochain complexes $(\Lambda(\mathfrak{A}_{n+1}), d)$ and $(\Lambda(\mathfrak{A}_{n+1}), \delta)$ respectively by (L_n, d) and (R_n, δ) .

Definition 3.1 For $n = 1, 2, \dots$, let G_n be the set of all triangular matrix $[a_{i,j}]$ with entries

$a_{i,j} = 0$ or 1

$$[a_{i,j}] = \begin{bmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ & & a_{2,3} & \cdots & a_{2,n} \\ & & & \cdots & \cdots \\ & & & & a_{n-1,n} \end{bmatrix}.$$

For a triangular matrix $[a_{i,j}] \in G_n$, its weight (i_0, i_1, \dots, i_n) is defined to be $i_s = \sum_{k=0}^{s-1} (1 - a_{k,s}) + \sum_{k=s+1}^n a_{s,k}$, $s = 0, 1, \dots, n$. $G(i_0, i_1, \dots, i_n)$ is defined to be the subset of G_n of all triangular matrices with weight (i_0, i_1, \dots, i_n) . Specifically, we define $G_0 = \{[]\}$ having only one ‘vacuum’ triangular matrix $[]$ with weight (0) .

We correspond a triangular matrix $[a_{i,j}]$ to a monomial $e_{0,1}^{a_{0,1}} e_{1,2}^{a_{1,2}} e_{0,2}^{a_{0,2}} \cdots e_{n-1,n}^{a_{n-1,n}} \cdots e_{1,n}^{a_{1,n}} e_{0,n}^{a_{0,n}}$ ($e_{i,j}^1 = e_{i,j}$, $e_{i,j}^0 = 1$) in $\Lambda(\mathfrak{A}_{n+1})$. With this correspondence, $G_n = \Lambda(\mathfrak{A}_{n+1})$. We denote by $G(i_0, i_1, \dots, i_n)$ the set of all triangular matrices with weight (i_0, i_1, \dots, i_n) and call it the weight subgraph of G_n . We denote the chain complex $(G(i_0, i_1, \dots, i_n), d)$ and the cochain complex $(G(i_0, i_1, \dots, i_n), \delta)$ respectively by $(L(i_0, i_1, \dots, i_n), d)$ and $(R(i_0, i_1, \dots, i_n), \delta)$ and call them the weight chain and cochain subcomplexes.

Remark. We use a triangular matrix $[a_{i,j}]$ to denote both the vertex of G_n and the product $\prod_{i,j} e_{i,j}^{a_{i,j}}$ in $\Lambda(\mathfrak{A}_{n+1})$ as defined in Definition 3.1. Since R_n is a DGA, it is sometimes convenient to discuss the problem on R_n but not on G_n or L_n . So we denote the algebra generator of R_n by $x_{i,j}$ to distinguish them from $e_{i,j}$ in G_n and L_n .

Theorem 3.1 The weight subgraphs satisfy the following.

1. Let S_{n+1} be the group of permutations on $\{0, 1, \dots, n\}$. For any $\sigma \in S_{n+1}$, there is a graph isomorphism $g(\sigma) : G_n \rightarrow G_n$ such that the restriction map $g(\sigma)|_{G(i_0, i_1, \dots, i_n)}$ is a graph isomorphism from $G(i_0, i_1, \dots, i_n)$ to $G(i_{\sigma(0)}, i_{\sigma(1)}, \dots, i_{\sigma(n)})$ which is generally not a GAD graph isomorphism.
2. For any weight $(i_0, i_1, \dots, i_{n-1}, i_n)$, there is a transpose GAD graph isomorphism from $G(i_0, i_1, \dots, i_{n-1}, i_n)$ to $G(n-i_n, n-i_{n-1}, \dots, n-i_1, n-i_0)$.
3. For any weight $(i_0, i_1, \dots, i_{n-1}, i_n)$, there is a rotation GAD graph isomorphism from $G(i_0, i_1, \dots, i_{n-1}, i_n)$ to $G(i_n, i_0, i_1, \dots, i_{n-1})$.

4. For any weight $(i_0, i_1, \dots, i_{n-1}, i_n)$, there is a duality GAD graph isomorphism from $G(i_0, i_1, \dots, i_{n-1}, i_n)$ to $G(n-i_0, n-i_1, \dots, n-i_{n-1}, n-i_n)$.

Proof. 1. For $k = 1, \dots, n$, let $\sigma_k = (k-1, k) \in S_{n+1}$ ($\sigma_k(k-1) = k$, $\sigma_k(k) = k-1$, $\sigma_k(i) = i$ for $i \neq k-1, k$). Define $g(\sigma_k)$ as follows. For any triangular matrices $[a_{i,j}] \in G_n$,

$$g(\sigma_k) \left(\begin{bmatrix} \cdots & a_{0,k-1} & a_{0,k} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{k-2,k-1} & a_{k-2,k} & \cdots & \cdots & \cdots \\ & & a_{k-1,k} & a_{k-1,k+1} & \cdots & a_{k-1,n} \\ & & & a_{k,k+1} & \cdots & a_{k,n} \\ & & & & \cdots & \cdots \end{bmatrix} \right) \\ = \begin{bmatrix} \cdots & a_{0,k} & a_{0,k-1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{k-2,k} & a_{k-2,k-1} & \cdots & \cdots & \cdots \\ & & 1 - a_{k-1,k} & a_{k,k+1} & \cdots & a_{k,n} \\ & & & a_{k-1,k+1} & \cdots & a_{k-1,n} \\ & & & & \cdots & \cdots \end{bmatrix},$$

where the omitted part remains unchanged. It is a direct checking that if a triangular matrix a has weight $(\dots, i_{k-1}, i_k, \dots)$, then $g(\sigma_k)(a)$ has weight $(\dots, i_k, i_{k-1}, \dots)$. Since $\sigma_1, \dots, \sigma_n$ generate S_{n+1} , $g(\sigma)$ is defined for all $\sigma \in S_{n+1}$.

2. Define DGA isomorphism $\varrho: R_n \rightarrow R_n$ by $\lambda(x_{i,j}) = x_{n-j,n-i}$. Then for any triangular matrix $[a_{i,j}] \in G_n$,

$$\varrho \left(\begin{bmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n-1} & a_{0,n} \\ & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ & & \cdots & \cdots & \cdots \\ & & & a_{n-2,n-1} & a_{n-2,n} \\ & & & & a_{n-1,n} \end{bmatrix} \right) \\ = \pm \begin{bmatrix} a_{n-1,n} & a_{n-2,n} & \cdots & a_{1,n} & a_{0,n} \\ & a_{n-2,n-1} & \cdots & a_{1,n-1} & a_{0,n-1} \\ & & \cdots & \cdots & \cdots \\ & & & a_{1,2} & a_{0,2} \\ & & & & a_{0,1} \end{bmatrix}$$

and ϱ induces a GAD graph isomorphism that sends $G(i_0, \dots, i_n)$ to $G(n-i_n, \dots, n-i_0)$.

3. Notice that R_n has two DGA-module structures over R_{n-1} . The first module structure is induced by R_{n-1} as a subalgebra and in this sense, R_n is freely generated by the set $\{1\} \cup \{x_{i_0,n} \cdots x_{i_s,n}, 0 \leq i_0 < \dots < i_s < n\}$. The second is induced by the monomorphism

$j: R_{n-1} \rightarrow R_n$ defined by $j(x_{i,j}) = x_{i+1,j+1}$ and in this sense, R_n is freely generated by the set $\{1\} \cup \{x_{0,i_0} \cdots x_{0,i_s}, 0 < i_0 < \cdots < i_s \leq n\}$. Define $\varpi: R_n \rightarrow R_n$ as follows. For any $a \in R_{n-1}$ and $0 \leq i_1 < \cdots < i_s < n$, $\varpi(ax_{i_0,n} \cdots x_{i_s,n}) = (-1)^{i_0+\cdots+i_s} j(a)x_{0,1} \cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,i_s+1} \cdots x_{0,n}$ (the hat represents cancelling the factor in the product $x_{0,1}x_{0,2} \cdots x_{0,n}$ and the term is abbreviated to $\cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,i_s+1} \cdots$ in the following formulas). Then,

$$\begin{aligned}
& \varpi(\delta(ax_{i_s,n} \cdots x_{i_0,n})) \\
&= \varpi((\delta a)(x_{i_s,n} \cdots x_{i_0,n})) \\
&\quad + \sum_{k,j} (-1)^{|a|+l-k} \varpi(ax_{i_k,j} x_{i_s,n} \cdots x_{i_{l-1},n} x_{j,n} x_{i_l,n} \cdots \hat{x}_{i_k,n} \cdots x_{i_0,n}) \\
&= (-1)^{i_0+\cdots+i_s} (j(\delta a))(\cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,i_s+1} \cdots) \\
&\quad + \sum_{k,j} (-1)^{|a|+l-k+i_0+\cdots+i_s+j-i_k} (j(ax_{i_k,j}))(\cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,j+1} \cdots x_{i_k,n} \cdots \hat{x}_{0,i_s+1} \cdots) \\
&= (-1)^{i_0+\cdots+i_s} (\delta(ja))(\cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,i_s+1} \cdots) \\
&\quad + \sum_{k,j} (-1)^{|a|+l-k+i_0+\cdots+i_s+j-i_k} (ja)(x_{i_k+1,j+1} \cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,j+1} \cdots x_{i_k,n} \cdots \hat{x}_{0,i_s+1} \cdots) \\
&= (-1)^{i_0+\cdots+i_s} (\delta(ja))(\cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,i_s+1} \cdots) \\
&\quad + \sum_{k,j} (-1)^{|a|+i_0+\cdots+i_s} (ja)(\delta(\cdots \hat{x}_{0,i_0+1} \cdots \hat{x}_{0,i_s+1} \cdots)) \\
&= \delta(\varpi(ax_{i_s,n} \cdots x_{i_0,n})).
\end{aligned}$$

Thus, ϖ is a DGA-module isomorphism satisfying that for any triangular matrix $[a_{i,j}] \in G_n$,

$$\begin{aligned}
& \varpi \left(\begin{bmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n-1} & a_{0,n} \\ & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ & & \cdots & \cdots & \cdots \\ & & & a_{n-2,n-1} & a_{n-2,n} \\ & & & & a_{n-1,n} \end{bmatrix} \right) \\
&= \pm \begin{bmatrix} 1-a_{0,n} & 1-a_{1,n} & 1-a_{2,n} & \cdots & 1-a_{n-1,n} \\ & a_{0,1} & a_{0,2} & \cdots & a_{0,n-1} \\ & & a_{1,2} & \cdots & a_{1,n-1} \\ & & & \cdots & \cdots \\ & & & & a_{n-2,n-1} \end{bmatrix}
\end{aligned}$$

and induces a GAD graph isomorphism that sends $G(i_0, \cdots, i_n)$ to $G(i_n, i_0, \cdots, i_{n-1})$.

4. Define duality isomorphism $\vartheta: R_n \rightarrow L_n$ as follows. For any $[a_{i,j}] \in G_n$,

$$\begin{aligned} & \vartheta \left(\begin{bmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n-1} & a_{0,n} \\ & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ & & \cdots & \cdots & \cdots \\ & & & a_{n-2,n-1} & a_{n-2,n} \\ & & & & a_{n-1,n} \end{bmatrix} \right) \\ = & (-1)^\tau \begin{bmatrix} 1-a_{0,1} & 1-a_{0,2} & \cdots & 1-a_{0,n-1} & 1-a_{0,n} \\ & 1-a_{1,2} & \cdots & 1-a_{1,n-1} & 1-a_{1,n} \\ & & \cdots & \cdots & \cdots \\ & & & 1-a_{n-2,n-1} & 1-a_{n-2,n} \\ & & & & 1-a_{n-1,n} \end{bmatrix}. \end{aligned}$$

where $\tau = \sum_{i,j} a_{i,j}[(1+2+\cdots+(j-1)) + (j-i-1)]$. It is obvious that $\vartheta\delta = d\vartheta$ and so ϑ induces a GAD graph isomorphism that sends $G(i_0, \dots, i_n)$ to $G(n-i_0, \dots, n-i_n)$. Q.E.D.

Remark The conclusion 1. and 4. of Theorem 3.1 can be generalized to all the semi-simple Lie algebras over \mathbb{C} . Precisely, the Weyl group of a positive system acts on the associated chain graph $\Lambda(\mathfrak{g})$ of the Lie algebra \mathfrak{g} generated by the positive system and there is Poncaré duality on it. This is a generalization of Kostant theorem from the complex number case to the ring of integers case. But other isomorphisms can not be naturally generalized to even diamond Lie algebras.

Definition 3.2 For $n \geq 0$, ω_n is the set of all $(n+1)$ -tuples (i_0, \dots, i_n) such that for all $0 \leq k_0 < \dots < k_s \leq n$, $i_{k_0} + i_{k_1} + \dots + i_{k_s} \geq 0 + 1 + \dots + s$ and $i_0 + i_1 + \dots + i_n = 0 + 1 + \dots + n$. For $s > 0$, $(i_0, \dots, i_s) \in \omega_s$ is reducible if there is $0 \leq m < s$ and $0 \leq u_0 < \dots < u_m \leq s$ such that $i_{u_0} + \dots + i_{u_m} = 0 + 1 + \dots + m$. Equivalently, $(i_0, \dots, i_s) \in \omega_s$ is reducible if it is a permutation of $(j_0, \dots, j_m, k_0+m+1, \dots, k_n+m+1)$ such that $(j_0, \dots, j_m) \in \omega_m$ and $(k_0, \dots, k_n) \in \omega_n$.

Theorem 3.2 $(i_0, \dots, i_n) \in \omega_n$ if and only if it is the weight of a non-empty weight subgraph $G(i_0, \dots, i_n)$.

Proof. We use induction on n . For $n = 0, 1$, the theorem is trivial. Suppose the theorem holds for $n > 1$. Then for $[a_{i,j}] \in G(i_0, \dots, i_n, i_{n+1})$, $i_{n+1} = \sum_{i=0}^n (1 - a_{i,n+1})$,

$$\begin{bmatrix} a_{0,1} & \cdots & a_{0,n} \\ & \cdots & a_{1,n} \\ & & \cdots \\ & & a_{n-1,n} \end{bmatrix} \in G(i_0 - a_{0,n+1}, i_1 - a_{1,n+1}, \dots, i_n - a_{n,n+1})$$

By the induction hypothesis, for all $0 \leq k_0 < \dots < k_s \leq n$, $(i_{k_0} - a_{k_0,n+1}) + \dots + (i_{k_s} - a_{k_s,n+1}) \geq 0 + 1 + \dots + s$. So

$$\begin{aligned} i_{k_0} + \dots + i_{k_s} &\geq 0 + 1 + \dots + s + \sum_{u=1}^s a_{k_u,n+1} \geq 0 + 1 + \dots + s, \\ i_{k_0} + \dots + i_{k_s} + i_{n+1} &\geq 0 + 1 + \dots + s + \sum_{u=1}^s a_{k_u,n+1} + \sum_{i=0}^n (1 - a_{i,n+1}) \geq 0 + 1 + \dots + s + (s+1), \\ i_0 + \dots + i_n + i_{n+1} &= i_0 + \dots + i_n + \sum_{i=0}^n a_{i,n+1} + \sum_{i=0}^n (1 - a_{i,n+1}) = 0 + 1 + \dots + n + (n+1). \end{aligned}$$

Thus, $(i_0, \dots, i_n, i_{n+1}) \in \omega_{n+1}$.

For $(i_0, \dots, i_n, i_{n+1}) \in \omega_{n+1}$, we will prove that $G(i_0, \dots, i_n, i_{n+1})$ is non-empty. By 1. of Theorem 3.1, we may suppose $i_0 \leq i_1 \leq \dots \leq i_n \leq i_{n+1}$. We use induction on $n+1-i_{n+1}$. If $i_{n+1} = n+1$, then $G(i_0, \dots, i_n, i_{n+1}) = G(i_0, \dots, i_n)$. The conclusion holds. Suppose for $(j_0, \dots, j_{n+1}) \in \omega_{n+1}$ and $j_{n+1} < n+1-s$, $G(j_0, \dots, j_{n+1})$ is non-empty. Then for $(i_0, \dots, i_n, i_{n+1}) \in \omega_{n+1}$ and $i_{n+1} = n+1-s$, let k be the biggest number such that $i_0 + i_1 + \dots + i_k = 0 + 1 + \dots + k$. Then $i_{k+1} > k+1$ and

$$\alpha = (i_0, \dots, i_k, i_{k+1}-1, \dots, i_{k+l}-1, i_{k+l+1}, \dots, i_n, i_{n+1}+l) \in \omega_{n+1} \quad (l = i_{k+1}-k-1).$$

By the induction hypothesis, there exists $[a_{i,j}] \in G(\alpha)$. We have $a_{k+1,n+1} = \dots = a_{k+l,n+1} = 0$. If not, then the product $e = \prod e_{i,j}^{a_{i,j}}$ has a factor $e_{k+i,n+1}$. We may suppose $i = 1$. Then the weight of $e/e_{k+1,n+1}$ is $(i_0, \dots, i_k, i_{k+1}-2, \dots, i_{k+l}-1, i_{k+l+1}, \dots, i_n, i_{n+1}+l+1)$. This $(n+1)$ -tuple is not in ω_{n+1} . A contradiction! So $ee_{k+1,n+1} \dots e_{k+l,n+1} \in G(i_0, \dots, i_n, i_{n+1})$. The induction is complete. Q.E.D.

Theorem 3.3 If (k_0, \dots, k_{m+n+1}) is reducible and is the permutation of $(i_0, \dots, i_m, j_0+m+1, \dots, j_n+m+1)$, then there is a GAD graph isomorphism

$$G(k_0, \dots, k_{m+n+1}) = G(i_0, \dots, i_m) \times G(j_0, \dots, j_n).$$

Specifically, for $m = 0$ and $n = 0$, we have

$$\begin{aligned} G(j_0+1, \dots, j_{k-1}+1, 0, j_k+1, \dots, j_n+1) &= G(j_0, \dots, j_{k-1}, j_k, \dots, j_n), \\ G(i_0, \dots, i_{k-1}, m+1, i_k, \dots, i_m) &= G(i_0, \dots, i_{k-1}, i_k, \dots, i_m). \end{aligned}$$

Proof. We consider the DGA (R_n, δ) . For a reducible weight (k_0, \dots, k_{m+n+1}) that is a permutation of $(i_0, \dots, i_m, j_0+m+1, \dots, j_n+m+1)$, there are injective order preserving maps $\sigma: \{0, \dots, m\} \rightarrow \{0, \dots, m+n+1\}$ and $\tau: \{0, \dots, n\} \rightarrow \{0, \dots, m+n+1\}$ such that $\text{im}\sigma \cup \text{im}\tau = \{0, \dots, m+n+1\}$, $\text{im}\sigma \cap \text{im}\tau = \emptyset$. Then, the two maps induce two algebra monomorphisms $\sigma_*: \Lambda(\mathfrak{A}_{m+1}) \rightarrow \Lambda(\mathfrak{A}_{m+n+1})$ and $\tau_*: \Lambda(\mathfrak{A}_{n+1}) \rightarrow \Lambda(\mathfrak{A}_{m+n+1})$ defined by $\sigma_*(x_{s,t}) = x_{\sigma(s), \sigma(t)}$ and $\tau_*(x_{s,t}) = x_{\tau(s), \tau(t)}$. σ_* and τ_* are not DGA homomorphisms. But let $c = \prod_{s,t} x_{\tau(s), \sigma(t)}$ (in any fixed order). Then, define linear map $\xi: R(i_0, \dots, i_m) \otimes R(j_0, \dots, j_n) \rightarrow R(k_0, \dots, k_{m+n+1})$ by that $\xi(a \otimes b) = \sigma_*(a)\tau_*(b)c$ for all $a \in R(i_0, \dots, i_m)$ and $b \in R(j_0, \dots, j_n)$. It is obvious that $\delta c = 0$ and $\delta(\sigma_*(a)\tau_*(b)c) = (\delta\sigma_*(a))\tau_*(b)c + (-1)^{|a|}\sigma_*(a)(\delta\tau_*(b))c$. Thus, ξ is a DGA monomorphism.

To prove that ξ is an epimorphism, we need only show the two free groups have the same dimension. By 1. of Theorem 3.1, we may suppose $(k_0, \dots, k_{m+n+1}) = (i_0, \dots, i_m, j_0+m+1, \dots, j_n+m+1)$. For $[a_{i,j}] \in R(i_0, \dots, i_m, j_0+m+1, \dots, j_n+m+1)$,

$$\begin{bmatrix} a_{0,1} & \cdots & a_{0,m+n} \\ & \cdots & a_{1,m+n} \\ & & \cdots \\ & & a_{m+n-1,m+n} \end{bmatrix} \in R(i_0-a_{0,m+n+1}, \dots, i_m-a_{m,m+n+1}, j_0-a_{m+1,m+n+1}, \dots, j_{n-1}-a_{m+n-1,m+n+1}).$$

Therefore, $i_0-a_{0,m+n+1} + \dots + i_m-a_{m,m+n+1} \geq \frac{m(m+1)}{2}$ and so $a_{k,m+n+1} = 0$ for $k = 0, \dots, m$. This implies that $R(i_0-a_{0,m+n+1}, \dots, i_m-a_{m,m+n+1}, j_0-a_{m+1,m+n+1}, \dots, j_{n-1}-a_{m+n-1,m+n+1})$ is also reducible and

$$\begin{bmatrix} a_{0,1} & \cdots & a_{0,m+n-1} \\ & \cdots & a_{1,m+n-1} \\ & & \cdots \\ & & a_{m+n-2,m+n-1} \end{bmatrix} \in R(i_0-a_{0,m+n}, \dots, i_m-a_{m,m+n}, j_{n-2}-a_{m+n-2,m+n}-a_{m+n-2,m+n+1}).$$

For the same reason, $a_{k,m+n} = 0$ for $k = 0, \dots, m$. Inductively, we have that $a_{k,m+l+1} = 0$ for $k = 0, \dots, m$ and $l = 0, \dots, n$. That is,

$$[a_{i,j}] = \begin{bmatrix} a_{0,1} & \cdots & a_{0,m} & 0 & 0 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & a_{m-1,m} & 0 & 0 & \cdots & 0 \\ & & & 0 & 0 & \cdots & 0 \\ & & & & a_{m+1,m+2} & \cdots & a_{m+1,m+n+1} \\ & & & & & \cdots & \cdots \\ & & & & & & a_{m+n,m+n+1} \end{bmatrix}.$$

So $\dim R(i_0, \dots, i_m, j_0+m+1, \dots, j_n+m+1) = \dim R(i_0, \dots, i_m) \times \dim R(j_0, \dots, j_n)$. Q.E.D.

Notice that the cohomology classes with reducible weight may not be decomposable. For example, the cohomology class of $H^*(R(i_0, \dots, i_n))$ with (i_0, \dots, i_n) a permutation of $(0, 1, \dots, n)$ is not always decomposable, but all such weights are reducible.

Theorem 3.4 For every $\alpha \in \omega_n$, $G(\alpha)$ is connected. Thus, ω_n is the set of connected components of the graph $G_n = \Lambda(\mathfrak{A}_{n+1})$. Moreover, if both $(i_0, \dots, i_s, \dots, i_t, \dots, i_n)$ and $(i_0, \dots, i_s+1, \dots, i_t-1, \dots, i_n)$ are in ω_n , then there is at least one $e \in G(i_0, \dots, i_n)$ such that $ee_{s,t} \in G(i_0, \dots, i_s+1, \dots, i_t-1, \dots, i_n)$.

Proof. Firstly, we prove the second conclusion. By 1. of Theorem 3.1, we need only prove the case that if both (i_0, i_1, \dots, i_n) and $(i_0+1, i_1-1, \dots, i_n)$ are in ω_n , then there is $e \in G(i_0, i_1, \dots, i_n)$ such that $ee_{0,1} \in G(i_0+1, i_1-1, \dots, i_n)$. Suppose $i_0 \leq i_1$. Let $e = [a_{i,j}] \in G(i_0, i_1, \dots, i_n)$. If $a_{0,1} = 0$, then $ee_{0,1} \in G(i_0+1, i_1-1, \dots, i_n)$. If $a_{0,1} = 1$, since $\sum_{i=2}^n a_{0,i} < \sum_{i=2}^n a_{0,i} + 1 \leq \sum_{i=2}^n a_{1,i}$, there is $2 \leq i \leq n$ such that $a_{1,i} = 1$ and $a_{0,i} = 0$. Let $e' = (e/e_{0,1}e_{1,i})e_{0,i}$, then $e' \in G(i_0, i_1, \dots, i_n)$ and $e'e_{0,1} \in G(i_0+1, i_1-1, \dots, i_n)$. Suppose $i_0 > i_1$. Firstly, apply the conclusion to $(n-i_0, n-i_1, \dots, n-i_n)$. Then apply the duality isomorphism ϑ in 4. of Theorem 3.1, we prove the case for $i_0 > i_1$.

Now we use induction on n to prove that for every $(i_0, \dots, i_n) \in \omega_n$, $G(i_0, \dots, i_n)$ is connected. For $n = 0, 1, 2$, it is a direct checking. Now suppose the conclusion holds for $n > 2$. Then for $(i_0, \dots, i_n, i_{n+1}) \in \omega_{n+1}$, we have a disjoint union of sets

$$G(i_0, \dots, i_n, i_{n+1}) = \bigcup_{\varepsilon_0 + \dots + \varepsilon_n = n+1-i_{n+1}} G(i_0-\varepsilon_0, \dots, i_n-\varepsilon_n) e_{0,n+1}^{\varepsilon_0} \cdots e_{n,n+1}^{\varepsilon_n},$$

where the union is taken throughout all $\varepsilon_s = 0$ or 1 such that $(i_0-\varepsilon_0, \dots, i_n-\varepsilon_n)$ is a weight. By the induction hypothesis, every $G(i_0-\varepsilon_0, \dots, i_n-\varepsilon_n)$ is connected in G_n and

so every $G(i_0 - \varepsilon_0, \dots, i_n - \varepsilon_n) e_{0,n+1}^{\varepsilon_0} \dots e_{n,n+1}^{\varepsilon_n}$ is connected in G_{n+1} . So we need only prove that these different subgraphs are joined by paths in G_{n+1} . Let $(i_0 - \varepsilon_0, \dots, i_n - \varepsilon_n)$ and $(i_0 - \varepsilon'_0, \dots, i_n - \varepsilon'_n)$ be weights of two different components of the above disjoint union such that $\varepsilon_s = \varepsilon'_s = 0$, $\varepsilon_t = \varepsilon'_t = 1$ for some $0 \leq s < t \leq n$, $\varepsilon_i = \varepsilon'_i$ for $i \neq s, t$. By the second conclusion of the theorem, there is $e \in G(i_0 - \varepsilon_0, \dots, i_n - \varepsilon_n)$ such that $ee_{s,t} \in G(i_0 - \varepsilon'_0, \dots, i_n - \varepsilon'_n)$. Then,

$$\begin{aligned} ee_{s,t} e_{0,n+1}^{\varepsilon'_0} \dots e_{n,n+1}^{\varepsilon'_n} &\in G(i_0 - \varepsilon'_0, \dots, i_n - \varepsilon'_n) e_{0,n+1}^{\varepsilon'_0} \dots e_{n,n+1}^{\varepsilon'_n} \\ ee_{0,n+1}^{\varepsilon_0} \dots e_{n,n+1}^{\varepsilon_n} &\in G(i_0 - \varepsilon_0, \dots, i_n - \varepsilon_n) e_{0,n+1}^{\varepsilon_0} \dots e_{n,n+1}^{\varepsilon_n} \end{aligned}$$

But $[ee_{s,t} e_{0,n+1}^{\varepsilon'_0} \dots e_{n,n+1}^{\varepsilon'_n}, ee_{0,n+1}^{\varepsilon_0} \dots e_{n,n+1}^{\varepsilon_n}]$ is an edge of G_{n+1} . Therefore, such two components of the above disjoint union can be joined by paths in G_{n+1} . This case implies that any two components of the above disjoint union can be joined by paths in G_{n+1} . So $G(i_0, \dots, i_n, i_{n+1})$ is connected. Q.E.D.

Theorem 3.5 The ranks of the weight subgraphs satisfy the formula

$$r(G(\dots, j+1, \dots, i-1, \dots)) - r(G(\dots, j, \dots, i, \dots)) = i - j - 1,$$

where the omitted parts of the two weights are the same. Thus, for a weight (i_0, i_1, \dots, i_n) such that $i_0 \leq i_1 \leq \dots \leq i_n$,

$$r(G(i_0, i_1, \dots, i_n)) = \sum_{k=0}^n m_k + \sum_{i_k < k} (k - i_k),$$

where $m_k = i_k + (i_k + 1) + \dots + (k - 1)$ if $i_k < k$; $m_k = -i_k - (i_k - 1) - \dots - (k + 1)$ if $i_k > k$; $m_k = 0$ if $i_k = k$.

Proof. Suppose j is the s -th number and i is the t -th number of the weight $(\dots j \dots i \dots)$. By the second conclusion of Theorem 3.4, there is $a = [a_{u,v}] \in G(\dots j \dots i \dots)$ such that $a_{s,t} = 0$ and $ae_{s,t} \in G(\dots j+1 \dots i-1 \dots)$. Let $b = [b_{u,v}]$ be the corresponding triangular matrix for $ae_{s,t}$. The following lists all the neighbors of a and b .

- (1) For $m < s$, $a_{m,s} = 0$, $a_{m,t} = 1$, let $a_m = [c_{u,v}]$ be defined as follows. $c_{m,s} = 1$, $c_{s,t} = 1$, $c_{m,t} = 0$, $c_{u,v} = a_{u,v}$, otherwise. Then a_m is a neighbor of a .
- (2) For $s < n < t$, $a_{s,n} = 1$, $a_{n,t} = 1$, let $a_n = [c_{u,v}]$ be defined as follows. $c_{s,n} = 0$, $c_{n,t} = 0$, $c_{s,t} = 1$, $c_{u,v} = a_{u,v}$, otherwise. Then a_n is a neighbor of a .
- (3) For $t < l$, $a_{s,l} = 1$, $a_{t,l} = 0$, let $a_l = [c_{u,v}]$ be defined as follows. $c_{s,l} = 0$, $c_{s,t} = 1$, $c_{t,l} = 1$, $c_{u,v} = a_{u,v}$, otherwise. Then a_l is a neighbor of a .

(4) For $m < s$, $b_{m,s} = 1$, $b_{m,t} = 0$, let $b_m = [c_{u,v}]$ be defined as follows. $c_{m,s} = 0$, $c_{s,t} = 0$, $c_{m,t} = 1$, $c_{u,v} = b_{u,v}$, otherwise. Then b_m is a neighbor of b .

(5) For $s < n < t$, $b_{s,n} = 0$, $b_{n,t} = 0$, let $b_n = [c_{u,v}]$ be defined as follows. $c_{s,n} = 1$, $c_{n,t} = 1$, $c_{s,t} = 0$, $c_{u,v} = b_{u,v}$, otherwise. Then b_n is a neighbor of b .

(6) For $t < l$, $b_{s,l} = 0$, $b_{t,l} = 1$, let $b_l = [c_{u,v}]$ be defined as follows. $c_{s,l} = 1$, $c_{s,t} = 0$, $c_{t,l} = 0$, $c_{u,v} = b_{u,v}$, otherwise. Then b_l is a neighbor of b .

Thus, $r(b) - r(a) =$ the number of b_k 's $-$ the number of a_k 's $= i - j - 1$.

Notice that $G(0, 1, \dots, n)$ has only one vertex, the triangular matrix with all entries 0 (1 of $\Lambda(\mathfrak{a}_{n+1})$). So $r(G(0, 1, \dots, n)) = 0$. Then the second formula is an easy induction on $r(G(i_0, \dots, i_n))$. Q.E.D.

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